

Perturbations around the zeros of classical orthogonal polynomials

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Abstract

Starting from degree \mathcal{N} solutions of a time dependent Schrödinger-like equation for classical orthogonal polynomials, a linear matrix equation describing perturbations around the \mathcal{N} zeros of the polynomial is derived. The matrix has remarkable Diophantine properties. Its eigenvalues are independent of the zeros. The corresponding eigenvectors provide the representations of the lower degree $(0, 1, \dots, \mathcal{N} - 1)$ polynomials in terms of the zeros of the degree \mathcal{N} polynomial. The results are valid universally for all the classical orthogonal polynomials, including the Askey scheme of hypergeometric orthogonal polynomials and its q -analogues.

1 Introduction

The properties of the zeros of orthogonal polynomials, in particular, of classical orthogonal polynomials have been a fascinating subject for many years [1, 2]. In this paper we define *classical orthogonal polynomials* as polynomials *satisfying second order differential or difference equations and the three term recurrence relations* [3]. That is, the Askey scheme of hypergeometric orthogonal polynomials and its q -analogues are included [4, 5, 6] but not the recently discovered multi-indexed [7] and exceptional [8] orthogonal polynomials. Here we report a small contribution to the subject by presenting an $\mathcal{N} \times \mathcal{N}$ matrix \mathcal{M} (3.9) possessing remarkable Diophantine properties. The matrix \mathcal{M} describes the small oscillations around the zeros of a degree \mathcal{N} classical orthogonal polynomial. Main results are (i) The eigenvalues of \mathcal{M} are independent of the zeros. They are the difference of those of the differential/difference operator $\tilde{\mathcal{H}}$ (2.5), which governs the classical orthogonal polynomial, corresponding to the degree \mathcal{N} and a lower degree, Theorem 3.1. (ii) The corresponding eigenvectors provide *representations of the lower degree polynomials in terms of the zeros of degree \mathcal{N} polynomial*, Theorem 3.2. The theorems are universal, meaning that they apply to all the classical orthogonal polynomials.

The idea of the present research was influenced by a recent paper of Bihun and Calogero [9] discussing the multi-particle interactions and their equilibrium implied by the classical orthogonal polynomials, in particular, the Wilson and the Racah polynomials.

This paper is organized as follows. In section two, we start with the time dependent Schrödinger equation (2.1) of quantum mechanics, which offers a unified framework for discussing the classical orthogonal polynomials as the main parts of the eigenfunctions. By restricting the general solution (2.3) to those up to degree \mathcal{N} , a time dependent equation (2.11) for degree \mathcal{N} polynomials is obtained in §2.1. Section 3 is the main part of the paper. By considering perturbations around the zeros of the degree \mathcal{N} classical polynomials, the time-dependent equation is rewritten as a linear equation with an $\mathcal{N} \times \mathcal{N}$ matrix \mathcal{M} describing the infinitesimal oscillations around the zeros. The main Theorems about the eigenvalues and eigenfunctions are stated as the natural consequence of the construction. In sections four, five and six, the various data of the classical orthogonal polynomials are presented for the explicit construction of the matrix \mathcal{M} for verification. Section four is for the Classical orthogonal polynomials, that is, the Hermite, Laguerre and Jacobi. Section five is for the $(q-)$ Askey scheme of hypergeometric polynomials having the continuous orthogonality weight functions, that is the Wilson and Askey-Wilson polynomials and their reduced form polynomials. Section six is for the $(q-)$ Askey scheme of hypergeometric polynomials having the purely discrete weight functions, that is the Racah and q -Racah polynomials and their reduced form polynomials. The final section is for a summary and comments. A small Appendix is for the symbols and definitions related to the $(q-)$ hypergeometric functions.

2 Time dependent Schrödinger equations

The time dependent Schrödinger equation

$$i \frac{\partial \Psi(x, t)}{\partial t} = \mathcal{H} \Psi(x, t) \quad (2.1)$$

is exactly solvable if the corresponding eigenvalue problem of the Hamiltonian or the Schrödinger operator \mathcal{H}

$$\mathcal{H} \phi_n(x) = \mathcal{E}(n) \phi_n(x), \quad n = 0, 1, \dots, \quad (2.2)$$

is exactly solvable. Throughout this paper the Hamiltonian \mathcal{H} is assumed to be time independent. In terms of the complete set of solutions $\{\mathcal{E}(n), \phi_n(x)\}$ of the eigenvalue problem

(2.2), the general solution of the time dependent Schrödinger equation is given by

$$\Psi(x, t) = \sum_{n=0}^{\infty} c_n e^{-i\mathcal{E}(n)t} \phi_n(x), \quad (2.3)$$

in which $\{c_n\}$ are the constants of integration.

Hereafter we discuss one dimensional exactly solvable Hamiltonian systems [10, 11, 12, 13, 14], with a Hamiltonian \mathcal{H} which is a second order differential or difference operator. Their eigenfunctions include *classical orthogonal polynomials*, *i.e.*, the Hermite, Laguerre, Jacobi, Wilson, Askey-Wilson, Racah and q -Racah polynomials, and others [7, 8]. In other words, all the hypergeometric orthogonal polynomials of Askey scheme constitute the main part of the eigenfunctions of certain ‘discrete’ quantum mechanical systems with pure imaginary and/or real shifts [12, 13, 14]. For most solvable examples, the eigenfunctions have a factorised form

$$\phi_n(x) = \phi_0(x) P_n(\eta(x)). \quad (2.4)$$

Here $\phi_0(x)$ is the ground state wave function and its square $\phi_0(x)^2$ provides the orthogonality weight function for the polynomial $P_n(\eta(x))$ of degree n^1 in $\eta(x)$, which is called the *sinusoidal coordinate* [15].

By similarity transforming the Hamiltonian in terms of the ground state wave function $\phi_0(x)$, we obtain the differential/difference operator $\tilde{\mathcal{H}}$:

$$\tilde{\mathcal{H}} \stackrel{\text{def}}{=} \phi_0(x)^{-1} \circ \mathcal{H} \circ \phi_0(x), \quad (2.5)$$

governing the classical polynomials $\{P_n(\eta(x))\}$:

$$\tilde{\mathcal{H}} P_n(\eta(x)) = \mathcal{E}(n) P_n(\eta(x)), \quad n = 0, 1, \dots \quad (2.6)$$

In other words, $\tilde{\mathcal{H}}$ keeps the polynomial space $\{1, P_1(\eta(x)), P_2(\eta(x)), \dots, P_n(\eta(x))\}$ invariant.

In the rest of this paper, we consider only those systems having the above factorised eigenfunctions (2.4). We further restrict our attention to the *classical orthogonal polynomials* $P_n(\eta(x))$, that is, the new (the multi-indexed [7] and exceptional [8]) orthogonal polynomials will not be included.

¹In the case of recently discovered *multi-indexed* orthogonal polynomials [7], n stands for the number of nodes in the orthogonality interval. The degree of the polynomial is greater than n .

2.1 Polynomial solutions

Let us fix a positive integer \mathcal{N} and restrict the general solution (2.3) to those having *degrees up to \mathcal{N}* :

$$\Psi_{\mathcal{N}}(x, t) = \sum_{n=0}^{\mathcal{N}} c_n e^{-i\mathcal{E}(n)t} \phi_n(x), \quad (2.7)$$

$$= e^{-i\mathcal{E}(\mathcal{N})t} \phi_0(x) \psi_{\mathcal{N}}(x, t). \quad (2.8)$$

Here the function

$$\psi_{\mathcal{N}}(x, t) \stackrel{\text{def}}{=} \sum_{n=0}^{\mathcal{N}} c_n e^{i(\mathcal{E}(\mathcal{N}) - \mathcal{E}(n))t} P_n(\eta(x)) \quad (2.9)$$

is a polynomial of degree \mathcal{N} in $\eta(x)$. We choose the coefficient $c_{\mathcal{N}}$ of the highest degree polynomial $P_{\mathcal{N}}(\eta(x))$ to make it monic:

$$c_{\mathcal{N}} P_{\mathcal{N}}(\eta(x)) = \prod_{n=1}^{\mathcal{N}} (\eta(x) - \eta(x_n)), \quad (2.10)$$

in which $\{\eta(x_n)\}$, $n = 1, \dots, \mathcal{N}$, are the *zeros* of $P_{\mathcal{N}}(\eta(x))$. The polynomial $\psi_{\mathcal{N}}(x, t)$ satisfies the time evolution equation

$$\frac{\partial \psi_{\mathcal{N}}(x, t)}{\partial t} = -i\tilde{\mathcal{H}}_{\mathcal{N}} \psi_{\mathcal{N}}(x, t), \quad \tilde{\mathcal{H}}_{\mathcal{N}} \stackrel{\text{def}}{=} \tilde{\mathcal{H}} - \mathcal{E}(\mathcal{N}). \quad (2.11)$$

For a given set of parameters $\{c_n\}$, $n = 0, \dots, \mathcal{N}-1$, the polynomial $\psi_{\mathcal{N}}(x, t)$ can be regarded as a t -dependent deformation of the highest degree monic polynomial $c_{\mathcal{N}} P_{\mathcal{N}}(\eta(x))$:

$$\psi_{\mathcal{N}}(x, t) = \prod_{n=1}^{\mathcal{N}} (\eta(x) - \eta(x_n(t))), \quad (2.12)$$

in which $\{x_n(t)\}$ are certain t -dependent functions, describing the zeros of $\psi_{\mathcal{N}}(x, t)$ at time t .

3 Perturbations around the zeros

Among the generic t -dependent deformations (2.12) of the classical polynomial $P_{\mathcal{N}}(\eta(x))$, let us focus on those describing *infinitesimal oscillations around the zeros* of $P_{\mathcal{N}}(\eta(x))$:

$$x_n(t) = x_n + \epsilon \gamma_n(t), \quad 0 < \epsilon \ll 1, \quad n = 1, \dots, \mathcal{N}. \quad (3.1)$$

In other words, instead of the general deformation (2.12) by $\{c_n\}$, we choose infinitesimal $\{c_n\}$ so that the deformation can be considered as perturbations around the zeros of $P_{\mathcal{N}}(\eta(x))$. The above ansatz (3.1) leads to

$$\psi_{\mathcal{N}}(x, t) = \prod_{n=1}^{\mathcal{N}} (\eta(x) - \eta(x_n)) - \epsilon \sum_{n=1}^{\mathcal{N}} \gamma_n(t) \dot{\eta}(x_n) \prod_{j \neq n}^{\mathcal{N}} (\eta(x) - \eta(x_j)) + O(\epsilon^2), \quad (3.2)$$

$$\text{with } \dot{\eta}(x) \stackrel{\text{def}}{=} \frac{d\eta(x)}{dx}. \quad (3.3)$$

With this ansatz, the l.h.s. of the time evolution equation (2.11) is a degree $\mathcal{N} - 1$ polynomial in $\eta(x)$:

$$- \epsilon \sum_{n=1}^{\mathcal{N}} \frac{d\gamma_n(t)}{dt} \dot{\eta}(x_n) \prod_{j \neq n}^{\mathcal{N}} (\eta(x) - \eta(x_j)) + O(\epsilon^2). \quad (3.4)$$

The r.h.s. is also a degree $\mathcal{N} - 1$ polynomial in $\eta(x)$:

$$i\epsilon \sum_{m=1}^{\mathcal{N}} \gamma_m(t) \dot{\eta}(x_m) \tilde{\mathcal{H}}_{\mathcal{N}} \prod_{j \neq m}^{\mathcal{N}} (\eta(x) - \eta(x_j)) + O(\epsilon^2), \quad (3.5)$$

since the leading polynomial $P_{\mathcal{N}}(\eta(x)) \propto \prod_{n=1}^{\mathcal{N}} (\eta(x) - \eta(x_n))$ is annihilated by $\tilde{\mathcal{H}}_{\mathcal{N}}$:

$$\tilde{\mathcal{H}}_{\mathcal{N}} P_{\mathcal{N}}(\eta(x)) = 0. \quad (3.6)$$

The polynomial evolution equation (2.11), being of an $\mathcal{N} - 1$ degree, is satisfied when its evaluation at \mathcal{N} independent points are satisfied and vice versa. Without loss of generality, we can choose the \mathcal{N} zeros $\{\eta(x_n)\}$ of $P_{\mathcal{N}}(\eta(x))$. This leads to \mathcal{N} linear ODE's for the unknown functions $\{\gamma_n(t)\}$ at the leading order of ϵ :

$$\begin{aligned} & \frac{d\gamma_n(t)}{dt} \dot{\eta}(x_n) \prod_{j \neq n}^{\mathcal{N}} (\eta(x_n) - \eta(x_j)) \\ &= -i \sum_{m=1}^{\mathcal{N}} \gamma_m(t) \dot{\eta}(x_m) \left(\tilde{\mathcal{H}}_{\mathcal{N}} \prod_{j \neq m}^{\mathcal{N}} (\eta(x) - \eta(x_j)) \right) \Big|_{x=x_n}, \quad n = 1, \dots, \mathcal{N}, \end{aligned} \quad (3.7)$$

which can be rewritten in a matrix form:

$$\frac{d\gamma_n(t)}{dt} = i \sum_{m=1}^{\mathcal{N}} \mathcal{M}_{nm} \gamma_m(t), \quad n = 1, \dots, \mathcal{N}, \quad (3.8)$$

$$\mathcal{M}_{nm} \stackrel{\text{def}}{=} - \frac{\dot{\eta}(x_m) \left(\tilde{\mathcal{H}}_{\mathcal{N}} \prod_{j \neq m}^{\mathcal{N}} (\eta(x) - \eta(x_j)) \right) \Big|_{x=x_n}}{\dot{\eta}(x_n) \prod_{j \neq n}^{\mathcal{N}} (\eta(x_n) - \eta(x_j))}. \quad (3.9)$$

By construction, we have the following

Theorem 3.1 *The eigenvalues of \mathcal{M} are*

$$\mathcal{E}(\mathcal{N}) - \mathcal{E}(m), \quad m = 0, 1, \dots, \mathcal{N} - 1, \quad (3.10)$$

which depend on the basic parameters of $\tilde{\mathcal{H}}$ but do not depend on the zeros $\{\eta(x_n)\}$ directly.

Theorem 3.2 *The corresponding eigenvectors $\{\mathbf{v}_n^{(m)}\}$ of \mathcal{M} ,*

$$\sum_{\ell=1}^{\mathcal{N}} \mathcal{M}_{n\ell} \mathbf{v}_\ell^{(m)} = (\mathcal{E}(\mathcal{N}) - \mathcal{E}(m)) \mathbf{v}_n^{(m)} \quad (3.11)$$

yield the representations of the lower degree polynomials $\{P_m(\eta)\}$, $m = 0, 1, \dots, \mathcal{N} - 1$, in terms of the zeros $\{\eta(x_n)\}$ of $P_{\mathcal{N}}(\eta(x))$:

$$\sum_{n=1}^{\mathcal{N}} \dot{\eta}(x_n) \mathbf{v}_n^{(m)} \prod_{j \neq n}^{\mathcal{N}} (\eta - \eta(x_j)) \propto P_m(\eta), \quad m = 0, 1, \dots, \mathcal{N} - 1. \quad (3.12)$$

The solution of the matrix equation (3.8), $\gamma_n^{(m)}(t) = e^{i(\mathcal{E}(\mathcal{N}) - \mathcal{E}(m))t} \mathbf{v}_n^{(m)}$ generates

$$\prod_{n=1}^{\mathcal{N}} (\eta(x) - \eta(x_n)) + \epsilon \alpha_m e^{i(\mathcal{E}(\mathcal{N}) - \mathcal{E}(m))t} P_m(\eta(x)) + O(\epsilon^2),$$

corresponding to $c_n \propto \epsilon \delta_{nm}$ in the polynomial solution (2.9). Here α_m is a certain constant. In other words, the present construction provides the expression of any *classical orthogonal polynomial* $P_m(\eta)$ in terms of the zeros of a higher degree polynomial $P_{\mathcal{N}}(\eta)$ of the same family.

The following Lemma is well known.

Lemma 3.3 *The Lagrangian interpolation of a polynomial $Q(x)$ ($\deg Q = m$) by a higher degree polynomial $\tilde{Q}(x)$, ($\deg \tilde{Q} = \mathcal{N} > m$) is exact:*

$$Q(x) = \sum_{n=1}^{\mathcal{N}} \frac{Q(x_n)}{\tilde{Q}'(x_n)} \cdot \left(\frac{\tilde{Q}(x)}{x - x_n} \right), \quad (3.13)$$

$$\tilde{Q}(x) \stackrel{\text{def}}{=} d_{\mathcal{N}} \prod_{n=1}^{\mathcal{N}} (x - x_n), \quad \tilde{Q}'(x_n) = d_{\mathcal{N}} \prod_{j \neq n}^{\mathcal{N}} (x_n - x_j). \quad (3.14)$$

In terms of the Lemma, Theorem 3.2 can be stated as

Corollary 3.4

$$\mathbf{v}_n^{(m)} \propto \frac{P_m(\eta(x_n))}{\dot{\eta}(x_n) P_{\mathcal{N}}'(\eta(x_n))} = \frac{P_m(\eta(x_n))}{\left(\frac{dP_{\mathcal{N}}(\eta(x))}{dx} \right) \Big|_{x=x_n}}, \quad n = 1, \dots, \mathcal{N},$$

$$m = 0, 1, \dots, \mathcal{N} - 1. \quad (3.15)$$

Remark 3.5 *This is rather remarkable. The matrix \mathcal{M} (3.9), constructed by the zeros of $P_N(\eta(x))$ and the basic parameters of $\tilde{\mathcal{H}}$ (2.11) only, contains all the information of the values of lower degree polynomials at these zeros $\{P_m(\eta(x_n))\}$, $m = 0, 1, \dots, N-1$ as eigenvectors.*

Remark 3.6 *The above eigenvalues (3.10) are algebraic numbers based on the basic parameters of $\tilde{\mathcal{H}}$ (2.5), N and the zeros, $\{\eta(x_n)\}$, $\{\dot{\eta}(x_n)\}$. The very fact that they are independent of the zeros means that the way that the zeros enter the matrix elements \mathcal{M}_{nm} is essential but that the explicit values of $\{\eta(x_n)\}$, $\{\dot{\eta}(x_n)\}$ are irrelevant. However, their explicit values are indispensable for the exact values of the eigenvectors $\{v_n^{(m)}\}$ to reproduce the lower degree polynomials $\{P_m(\eta)\}$ (3.12). The algebraic equations satisfied by the zeros play essential roles. They are simply obtained by evaluating the polynomial equation (3.6) at the zeros:*

$$0 = \tilde{\mathcal{H}}_N P_N(\eta(x)) \Big|_{x=x_n}. \quad (3.16)$$

For the Hermite, Laguerre and Jacobi polynomials, these are well known, see (4.4), (4.12) and (4.24). The matrix elements of \mathcal{M} (3.9) are very closely related with them.

Remark 3.7 *The matrix \mathcal{M} (3.9) is conceptually and structurally much simpler than the related matrices introduced for the Hermite, Laguerre and Jacobi polynomials by Ahmed et al [16] and for the Wilson and Racah by Bihun-Calogero [9]. The corresponding matrices have the same eigenvalues (up to an additive constant and an overall factor). As will be shown in §4, the matrices in [16] for the Hermite, Laguerre and Jacobi polynomials and the matrix \mathcal{M} in Theorem 3.2 share the same eigenvectors (4.7),(4.15),(4.27). Based on this fact we demonstrate explicitly that the eigenvectors $\{v_n^{(m)}\}$ of \mathcal{M} have the above form (3.15) for the Classical orthogonal polynomials.*

In the subsequent sections, we provide explicit examples and data of the classical orthogonal polynomials for which the above Theorems apply. The Hermite, Laguerre and Jacobi polynomials in section four. They are the main part of the eigenfunctions of exactly solvable systems in ordinary quantum mechanics. The Wilson and Askey-Wilson polynomials and their reduced form polynomials are discussed in section five. They are the main part of the eigenfunctions of exactly solvable systems in *discrete quantum mechanics with pure imaginary shifts* [13, 14]. The Racah and q -Racah polynomials and their reduced form polynomials are examined in section six. They are the main part of the eigenfunctions of exactly solvable systems in *discrete quantum mechanics with real shifts* [12, 14]. The examples in

sections five and six are grouped according to the sinusoidal coordinates. In section five, the group of linear in x , $\eta(x) = x$, contains the continuous Hahn and Meixner-Pollaczek. The group of quadratic in x , $\eta(x) = x^2$, consists of the Wilson and continuous Hahn. The group of $\eta(x) = \cos x$ comprises of the Askey-Wilson, continuous dual q -Hahn, Al-Salam-Chihara, continuous (big) q -Hermite, and continuous q -Jacobi (Laguerre). In section six, the group of linear in x , $\eta(x) = x$, contains the Hahn, Krawtchouk, Meixner and Charlier. The group of quadratic in x , $\eta(x) = x(x + d)$, consists of the Racah and dual Hahn. The group of linear in $q^{\pm x}$, $\eta(x) = q^{-x} - 1, 1 - q^x$, contains the q -Hahn, (quantum, affine) q -Krawtchouk, little q -Jacobi, q -Meixner, little q -Laguerre, Al-Salam-Carlitz II, and (alternative) q -Charlier. The group of 'bilinear' in $q^{\pm x}$ consists of the q -Racah and dual q -Hahn. For each example, we will provide the explicit form of the polynomial $\{P_n(\eta)\}$, the sinusoidal coordinate $\eta(x)$ and the second order differential/difference operator $\tilde{\mathcal{H}}$, so that self contained verification of the Theorems could be made, either algebraically or numerically. In these examples, we adopt the following short hand notation

$$y_n \equiv \eta(x_n), \quad n = 1, \dots, \mathcal{N}, \quad (3.17)$$

for notational simplicity, except for the linear cases, $\eta(x) = x$.

4 Examples from ordinary quantum mechanics

4.1 Hermite

The system has no parameter and the various data are:

$$\tilde{\mathcal{H}} = -\frac{d^2}{dx^2} + 2x\frac{d}{dx}, \quad -\infty < x < \infty, \quad \eta(x) = x, \quad \mathcal{E}(n) = 2n, \quad (4.1)$$

$$\phi_0(x)^2 = e^{-x^2}, \quad P_n(\eta) = H_n(\eta), \quad \text{Hermite polynomial.} \quad (4.2)$$

The matrix \mathcal{M} (3.9) reads

$$\begin{aligned} \mathcal{M}_{nm} = 2\delta_{nm} & \left(\mathcal{N} + \sum'_{j < k} \frac{1}{x_n - x_j} \cdot \frac{1}{x_n - x_k} - x_n \sum'_{j=1}^{\mathcal{N}} \frac{1}{x_n - x_j} \right) \\ & + 2(1 - \delta_{nm}) \frac{1}{x_n - x_m} \left(\sum'_{j=1, \neq m}^{\mathcal{N}} \frac{1}{x_n - x_j} - x_n \right). \end{aligned} \quad (4.3)$$

A prime appended to a sum indicates that the singular terms are omitted. This matrix has a remarkable Diophantine property. It is elementary to verify for lower \mathcal{N} that the eigenvalues

of \mathcal{M} are all integers

$$2(\mathcal{N} - m), \quad m = 0, 1, \dots, \mathcal{N} - 1,$$

for *arbitrary distinct complex numbers* $\{x_n\}$.

The polynomial equation for $H_{\mathcal{N}}$ (3.16) yields the well-known algebraic equations among the zeros $\{x_n\}$:

$$\sum_{j=1}^{\mathcal{N}}{}' \frac{1}{x_n - x_j} = x_n. \quad (4.4)$$

In terms of the zeros $\{x_n\}$ of $H_{\mathcal{N}}$, $H_{\mathcal{N}}(x_n) = 0$, Ahmed et al [16] introduced an $\mathcal{N} \times \mathcal{N}$ matrix

$$A_{nm} = \delta_{nm} \sum_{j=1}^{\mathcal{N}}{}' \frac{1}{(x_n - x_j)^2} - (1 - \delta_{nm}) \frac{1}{(x_n - x_m)^2}, \quad (4.5)$$

having the eigenvector

$$v_n^{(m)} = \frac{H_m(x_n)}{H_{\mathcal{N}-1}(x_n)}, \quad n = 1, \dots, \mathcal{N}, \quad (4.6)$$

corresponding to the eigenvalue $\mathcal{N} - m - 1$, $m = 0, 1, \dots, \mathcal{N} - 1$. It is elementary to show

$$\mathcal{M} = 2(A + 1) \quad (4.7)$$

by using (4.4) and another equation

$$\sum_{j=1}^{\mathcal{N}}{}' \frac{1}{(x_n - x_j)^2} = \frac{2}{3}(\mathcal{N} - 1) - \frac{1}{3}x_n^2. \quad (4.8)$$

This provides the direct derivation of Corollary 3.4 (3.15), since $H'_{\mathcal{N}}(x) = 2H_{\mathcal{N}-1}(x)$.

4.2 Laguerre

The system has one parameter $g > -\frac{1}{2}$ and the various data are:

$$\tilde{\mathcal{H}} = -\frac{d^2}{dx^2} + 2\left(x - \frac{g}{x}\right)\frac{d}{dx}, \quad 0 < x < \infty, \quad \eta(x) = x^2, \quad \mathcal{E}(n) = 4n, \quad (4.9)$$

$$\phi_0(x)^2 = e^{-x^2}(x^2)^g, \quad P_n(\eta) = L_n^{(\alpha)}(\eta), \quad \text{Laguerre polynomial,} \quad \alpha \stackrel{\text{def}}{=} g - \frac{1}{2}. \quad (4.10)$$

The matrix \mathcal{M} (3.9) reads ($y_n = x_n^2$)

$$\begin{aligned} \mathcal{M}_{nm} = & 4\delta_{nm} \left(\mathcal{N} + 2y_n \sum_{j < k}^{\mathcal{N}}{}' \frac{1}{y_n - y_j} \cdot \frac{1}{y_n - y_k} - (y_n - \alpha - 1) \sum_{j=1}^{\mathcal{N}}{}' \frac{1}{y_n - y_j} \right) \\ & + 4(1 - \delta_{nm}) \frac{x_m}{x_n} \cdot \frac{1}{y_n - y_m} \left(2y_n \sum_{j=1, \neq m}^{\mathcal{N}}{}' \frac{1}{y_n - y_j} - (y_n - \alpha - 1) \right). \end{aligned} \quad (4.11)$$

This matrix has a remarkable Diophantine property. It is elementary to verify for lower \mathcal{N} that the eigenvalues of \mathcal{M} are all integers

$$4(\mathcal{N} - m), \quad m = 0, 1, \dots, \mathcal{N} - 1,$$

for *arbitrary distinct complex numbers* $\{y_n\}$ and $\{x_n\}$ except for 0.

The polynomial equation for $L_{\mathcal{N}}^{(\alpha)}$ (3.16) with a change of variables yields the well-known algebraic equations among the zeros $\{y_n\}$:

$$y_n \sum_{j=1}^{\mathcal{N}} \frac{1}{y_n - y_j} = \frac{1}{2}(y_n - (\alpha + 1)). \quad (4.12)$$

In terms of the zeros $\{y_n\}$ of $L_{\mathcal{N}}^{(\alpha)}$, $L_{\mathcal{N}}^{(\alpha)}(y_n) = 0$, Ahmed et al [16] introduced an $\mathcal{N} \times \mathcal{N}$ matrix

$$B_{nm} = \delta_{nm} \sum_{j=1}^{\mathcal{N}} \frac{y_j}{(y_n - y_j)^2} - (1 - \delta_{nm}) \frac{y_m}{(y_n - y_m)^2}, \quad (4.13)$$

having the eigenvector

$$v_n^{(m)} = \frac{L_m^{(\alpha)}(y_n)}{L_{\mathcal{N}-1}^{(\alpha)}(y_n)}, \quad n = 1, \dots, \mathcal{N}, \quad (4.14)$$

corresponding to the eigenvalue $\frac{1}{2}(\mathcal{N} - m - 1)$, $m = 0, 1, \dots, \mathcal{N} - 1$. It is elementary to show

$$\mathcal{M} = 4\mathcal{D}(2B + 1)\mathcal{D}^{-1}, \quad \mathcal{D} \stackrel{\text{def}}{=} \text{diag}(x_1, x_2, \dots, x_{\mathcal{N}}), \quad (4.15)$$

by using (4.12) and another equation [16]

$$y_n^2 \sum_{j=1}^{\mathcal{N}} \frac{1}{(y_n - y_j)^2} = -\frac{1}{12} ((\alpha + 1)(\alpha + 5) - 2(2\mathcal{N} + \alpha + 1)y_n + y_n^2). \quad (4.16)$$

This means that the eigenvectors of \mathcal{M} (4.11) are

$$v_n^{(m)} = x_n \frac{L_m^{(\alpha)}(y_n)}{L_{\mathcal{N}-1}^{(\alpha)}(y_n)} \propto \frac{1}{x_n} \frac{L_m^{(\alpha)}(y_n)}{L_{\mathcal{N}}^{(\alpha)}(y_n)}, \quad m = 0, 1, \dots, \mathcal{N} - 1, \quad (4.17)$$

providing the direct derivation of Corollary 3.4 (3.15). In deriving the final proportionality relation, the following identity ((5.1.14) of [2]) is useful:

$$\eta \frac{dL_n^{(\alpha)}(\eta)}{d\eta} = -\eta L_{n-1}^{(\alpha+1)}(\eta) = nL_n^{(\alpha)}(\eta) - (n + \alpha)L_{n-1}^{(\alpha)}(\eta). \quad (4.18)$$

4.3 Jacobi

The system has two parameters $g > -\frac{1}{2}$, $h > -\frac{1}{2}$ and the various data are:

$$\tilde{\mathcal{H}} = -\frac{d^2}{dx^2} - 2(g \cot x - h \tan x) \frac{d}{dx}, \quad 0 < x < \frac{\pi}{2}, \quad (4.19)$$

$$\eta(x) = \cos 2x, \quad \dot{\eta}(x) = -2 \sin 2x, \quad (\dot{\eta}(x))^2 = 4(1 - \eta(x)^2), \quad (4.20)$$

$$\phi_0(x)^2 = (\sin^2 x)^g (\cos^2 x)^h, \quad P_n(\eta) = P_n^{(\alpha, \beta)}(\eta), \quad \text{Jacobi polynomial}, \quad (4.21)$$

$$\mathcal{E}(n) = 4n(n + g + h) = 4n(n + \alpha + \beta + 1), \quad \alpha \stackrel{\text{def}}{=} g - \frac{1}{2}, \quad \beta \stackrel{\text{def}}{=} h - \frac{1}{2}. \quad (4.22)$$

The matrix \mathcal{M} (3.9) reads ($y_n = \cos 2x_n$)

$$\begin{aligned} \mathcal{M}_{nm} = 4\delta_{nm} & \left(\mathcal{N}(\mathcal{N} + \alpha + \beta + 1) + 2(1 - y_n^2) \sum_{j < k}^{\mathcal{N}} \frac{1}{y_n - y_j} \cdot \frac{1}{y_n - y_k} \right. \\ & \left. - ((\alpha + \beta)y_n + \alpha - \beta) \sum_{j=1}^{\mathcal{N}} \frac{1}{y_n - y_j} \right) \\ & + 4(1 - \delta_{nm}) \frac{\sin 2x_m}{\sin 2x_n} \cdot \frac{1}{y_n - y_m} \left(2(1 - y_n^2) \sum_{j=1, \neq m}^{\mathcal{N}} \frac{1}{y_n - y_j} - ((\alpha + \beta)y_n + \alpha - \beta) \right). \end{aligned} \quad (4.23)$$

This matrix has a remarkable Diophantine property when $\alpha + \beta$ is an integer. It is elementary to verify for lower \mathcal{N} that the eigenvalues of \mathcal{M} (4.23) are

$$4(\mathcal{N} - m)(\mathcal{N} + m + \alpha + \beta + 1), \quad m = 0, 1, \dots, \mathcal{N} - 1,$$

for *arbitrary distinct complex numbers* $\{y_n\}$ and $\{x_n\}$ except for $0 \bmod \pi/2$.

The polynomial equation for $P_{\mathcal{N}}^{(\alpha, \beta)}$ (3.16) with a change of variables yields the well-known algebraic equations among the zeros $\{y_n\}$:

$$(1 - y_n^2) \sum_{j=1}^{\mathcal{N}} \frac{1}{y_n - y_j} = \frac{1}{2}((\alpha + \beta + 2)y_n + \alpha - \beta). \quad (4.24)$$

In terms of the zeros $\{y_n\}$ of $P_{\mathcal{N}}^{(\alpha, \beta)}$, $P_{\mathcal{N}}^{(\alpha, \beta)}(y_n) = 0$, Ahmed et al [16] introduced an $\mathcal{N} \times \mathcal{N}$ matrix

$$C_{nm} = \delta_{nm} \sum_{j=1}^{\mathcal{N}} \frac{(1 - y_j^2)}{(y_n - y_j)^2} - (1 - \delta_{nm}) \frac{(1 - y_m^2)}{(y_n - y_m)^2}, \quad (4.25)$$

having the eigenvector

$$v_n^{(m)} = \frac{P_m^{(\alpha, \beta)}(y_n)}{P_{\mathcal{N}-1}^{(\alpha, \beta)}(y_n)}, \quad n = 1, \dots, \mathcal{N}, \quad (4.26)$$

corresponding to the eigenvalue $\frac{1}{2}(\mathcal{N} - m - 1)(\mathcal{N} + m + \alpha + \beta)$, $m = 0, 1, \dots, \mathcal{N} - 1$. It is elementary to show

$$\mathcal{M} = 4\mathcal{D}(2C + 2\mathcal{N} + \alpha + \beta)\mathcal{D}^{-1}, \quad \mathcal{D} \stackrel{\text{def}}{=} \text{diag}(\sin 2x_1, \sin 2x_2, \dots, \sin 2x_{\mathcal{N}}), \quad (4.27)$$

by using (4.24) and another equation [16]

$$\begin{aligned} & (1 - y_n^2)^2 \sum_{j=1}^{\mathcal{N}} \frac{1}{(y_n - y_j)^2} \\ &= \frac{1}{3}(\mathcal{N} - 1)(\mathcal{N} + \alpha + \beta + 2) - \frac{1}{12}(\alpha - \beta)^2 - \frac{1}{6}(\alpha - \beta)(\alpha + \beta + 6)y_n \\ & \quad - \frac{1}{12}[4\mathcal{N}(\mathcal{N} + \alpha + \beta + 1) + (\alpha + \beta + 2)(\alpha + \beta + 6)]y_n^2. \end{aligned} \quad (4.28)$$

This means that the eigenvectors of \mathcal{M} (4.23) are

$$\mathbf{v}_n^{(m)} = \sin 2x_n \frac{P_m^{(\alpha, \beta)}(y_n)}{P_{\mathcal{N}-1}^{(\alpha, \beta)}(y_n)} \propto \frac{1}{\sin 2x_n} \frac{P_m^{(\alpha, \beta)}(y_n)}{P_{\mathcal{N}}^{(\alpha, \beta)'}(y_n)}, \quad m = 0, 1, \dots, \mathcal{N} - 1, \quad (4.29)$$

providing the direct derivation of Corollary 3.4 (3.15). In deriving the final proportionality relation, the following identity ((4.5.7) of [2]) is useful:

$$\begin{aligned} & (2n + \alpha + \beta)(1 - \eta^2) \frac{dP_n^{(\alpha, \beta)}(\eta)}{d\eta} \\ &= -n[(2n + \alpha + \beta)\eta + \beta - \alpha] P_n^{(\alpha, \beta)}(\eta) + 2(n + \alpha)(n + \beta) P_{n-1}^{(\alpha, \beta)}(\eta). \end{aligned} \quad (4.30)$$

5 Examples from discrete quantum mechanics with pure imaginary shifts

The difference operator $\tilde{\mathcal{H}}$ governing the classical orthogonal polynomials (2.6) belonging to this class depends on an analytic function $V(x)$ of x :

$$\tilde{\mathcal{H}} = V(x)(e^{-i\gamma\partial_x} - 1) + V^*(x)(e^{i\gamma\partial_x} - 1). \quad (5.1)$$

Throughout this section we use $i \stackrel{\text{def}}{=} \sqrt{-1}$. The function $V^*(x)$ is an analytic function of x obtained from $V(x)$ by the $*$ -operation, which is defined as follows. If $f(x) = \sum_n a_n x^n$, $a_n \in \mathbb{C}$, then $f^*(x) \stackrel{\text{def}}{=} \sum_n a_n^* x^n$, in which a_n^* is the complex conjugation of a_n . Here γ is a real parameter specifying the shifts of the functions:

$$e^{\pm i\gamma\partial_x} \psi(x) = \psi(x \pm i\gamma).$$

We consider two different cases $\gamma = 1$ for the Wilson polynomials and its reduced form polynomials and $\gamma = \log q$ and $0 < q < 1$ for the Askey-Wilson polynomials and its reduced form polynomials. The polynomials are assembled into three groups according to the form of the sinusoidal coordinate $\eta(x) = x, x^2$ and $\cos x$. In each group, we start from the most generic member and move to simpler ones. For a comprehensive exposition of these polynomials in the discrete quantum mechanics formulation, see [13, 14].

5.1 Polynomials having $\eta(x) = x$, $-\infty < x < \infty$, $\gamma = 1$

This group consists of two members, the continuous Hahn §5.1.1 and the Meixner-Pollaczek §5.1.2. The matrix \mathcal{M} (3.9) for this group reads

$$\begin{aligned} \mathcal{M}_{nm} = \delta_{nm} & \left(-\frac{V(x_n) \prod_{j \neq n}^{\mathcal{N}} (x_n - i - x_j) + V^*(x_n) \prod_{j \neq n}^{\mathcal{N}} (x_n + i - x_j)}{\prod_{j \neq n}^{\mathcal{N}} (x_n - x_j)} \right. \\ & \left. + \mathcal{E}(\mathcal{N}) + V(x_n) + V^*(x_n) \right) \\ & + (1 - \delta_{nm}) \frac{i \left(V(x_n) \prod_{j \neq n, m}^{\mathcal{N}} (x_n - i - x_j) - V^*(x_n) \prod_{j \neq n, m}^{\mathcal{N}} (x_n + i - x_j) \right)}{\prod_{j \neq n}^{\mathcal{N}} (x_n - x_j)}. \end{aligned} \quad (5.2)$$

It has eigenvalues $\mathcal{E}(\mathcal{N}) - \mathcal{E}(m)$, $m = 0, 1, \dots, \mathcal{N} - 1$, for arbitrary distinct complex values of $\{x_n\}$. For the zeros $\{x_n\}$ of $P_{\mathcal{N}}$, $P_{\mathcal{N}}(x_n) = 0$, it is straightforward to verify Theorem 3.2 numerically for lower \mathcal{N} . That is, the eigenvectors of the matrix \mathcal{M} (5.2) generate the lower degree polynomials $\{P_m(x)\}$, $m = 0, 1, \dots, \mathcal{N} - 1$ as in (3.12).

5.1.1 continuous Hahn

This polynomial depends on two complex parameters (a_1, a_2) and the various data are [6, 13]:

$$V(x) = (a_1 + ix)(a_2 + ix), \quad V^*(x) = (a_3 - ix)(a_4 - ix), \quad (5.3)$$

$$\mathcal{E}(n) = n(n + b_1 - 1), \quad b_1 \stackrel{\text{def}}{=} \sum_{j=1}^4 a_j, \quad \{a_3, a_4\} = \{a_1^*, a_2^*\} \text{ as a set}, \quad \text{Re } a_j > 0, \quad (5.4)$$

$$\phi_0(x)^2 = \prod_{j=1}^2 \Gamma(a_j + ix) \Gamma(a_j^* - ix), \quad (5.5)$$

$$P_n(x) = i^n \frac{(a_1 + a_3)_n (a_1 + a_4)_n}{n!} {}_3F_2 \left(\begin{matrix} -n, n + b_1 - 1, a_1 + ix \\ a_1 + a_3, a_1 + a_4 \end{matrix} \middle| 1 \right). \quad (5.6)$$

The polynomial equation for $P_{\mathcal{N}}$ (3.16) provides algebraic equations for the zeros $\{x_n\}$:

$$(a_1 + ix_n)(a_2 + ix_n) \prod_{j \neq n}^{\mathcal{N}} (x_n - i - x_j) = (a_3 - ix_n)(a_4 - ix_n) \prod_{j \neq n}^{\mathcal{N}} (x_n + i - x_j). \quad (5.7)$$

5.1.2 Meixner-Pollaczek

This polynomial depends on two real parameters (a, ϕ) and the various data are [6, 13]:

$$V(x) \stackrel{\text{def}}{=} e^{i(\frac{\pi}{2}-\phi)}(a + ix), \quad V^*(x) = e^{-i(\frac{\pi}{2}-\phi)}(a - ix), \quad (5.8)$$

$$\mathcal{E}(n) = 2n \sin \phi, \quad a > 0, \quad 0 < \phi < \pi \quad (5.9)$$

$$\phi_0(x)^2 = e^{(2\phi-\pi)x} \Gamma(a + ix) \Gamma(a - ix), \quad (5.10)$$

$$P_n(x) = \frac{(2a)_n}{n!} e^{in\phi} {}_2F_1\left(\begin{matrix} -n, a + ix \\ 2a \end{matrix} \middle| 1 - e^{-2i\phi}\right). \quad (5.11)$$

The polynomial equation for $P_{\mathcal{N}}$ (3.16) provides algebraic equations for the zeros $\{x_n\}$:

$$e^{i(\frac{\pi}{2}-\phi)x} (a + ix_n) \prod_{j \neq n}^{\mathcal{N}} (x_n - i - x_j) = e^{-i(\frac{\pi}{2}-\phi)x} (a - ix_n) \prod_{j \neq n}^{\mathcal{N}} (x_n + i - x_j), \quad (5.12)$$

which simplifies for $\phi = \frac{\pi}{2}$.

5.2 Polynomials having $\eta(x) = x^2$, $0 < x < \infty$, $\gamma = 1$

This group consists of two members, the Wilson §5.2.1 and the continuous dual Hahn §5.2.2.

The matrix \mathcal{M} (3.9) for this group reads

$$\begin{aligned} \mathcal{M}_{nm} = \delta_{nm} \left(-\frac{V(x_n) \prod_{j \neq n}^{\mathcal{N}} ((x_n - i)^2 - y_j) + V^*(x_n) \prod_{j \neq n}^{\mathcal{N}} ((x_n + i)^2 - y_j)}{\prod_{j \neq n}^{\mathcal{N}} (y_n - y_j)} \right. \\ \left. + \mathcal{E}(\mathcal{N}) + V(x_n) + V^*(x_n) \right) \\ + (1 - \delta_{nm}) \frac{x_m}{x_n} \frac{1}{\prod_{j \neq n}^{\mathcal{N}} (y_n - y_j)} \left(V(x_n) (1 + 2ix_n) \prod_{j \neq n, m}^{\mathcal{N}} ((x_n - i)^2 - y_j) \right. \\ \left. + V^*(x_n) (1 - 2ix_n) \prod_{j \neq n, m}^{\mathcal{N}} ((x_n + i)^2 - y_j) \right). \quad (5.13) \end{aligned}$$

It has eigenvalues $\mathcal{E}(\mathcal{N}) - \mathcal{E}(m)$, $m = 0, 1, \dots, \mathcal{N} - 1$, for arbitrary distinct complex values of $\{x_n\}$ except for the poles of V and V^* and $\{y_n = x_n^2\}$. They are all integers for the continuous dual Hahn. The same for the Wilson, if b_1 (5.19) is an integer. For the zeros $\{y_n = x_n^2\}$ of

$P_N, P_N(y_n) = 0$, it is straightforward to verify Theorem 3.2 numerically for lower \mathcal{N} . That is, the eigenvectors of the matrix \mathcal{M} (5.13) generate the lower degree polynomials $\{P_m(\eta)\}$, $m = 0, 1, \dots, \mathcal{N} - 1$ as in (3.12).

5.2.1 Wilson

This polynomial depends on four real parameters or two complex parameters, $\{a_1^*, a_2^*, a_3^*, a_4^*\} = \{a_1, a_2, a_3, a_4\}$ as a set, and the various data are [6, 13]:

$$V(x) = \frac{(a_1 + ix)(a_2 + ix)(a_3 + ix)(a_4 + ix)}{2ix(2ix + 1)}, \quad V^*(x) = V(-x), \quad \text{Re } a_j > 0, \quad (5.14)$$

$$\phi_0(x)^2 = \frac{\prod_{j=1}^4 \Gamma(a_j + ix)\Gamma(a_j - ix)}{\Gamma(2ix)\Gamma(-2ix)}, \quad \mathcal{E}(n) = n(n + b_1 - 1), \quad b_1 \stackrel{\text{def}}{=} \sum_{j=1}^4 a_j, \quad (5.15)$$

$$P_n(\eta(x)) = (a_1 + a_2)_n(a_1 + a_3)_n(a_1 + a_4)_n \times {}_4F_3\left(\begin{matrix} -n, n + b_1 - 1, a_1 + ix, a_1 - ix \\ a_1 + a_2, a_1 + a_3, a_1 + a_4 \end{matrix} \middle| 1\right). \quad (5.16)$$

The polynomial equation for P_N (3.16) provides algebraic equations for the zeros $\{y_n\}$:

$$\prod_{k=1}^4 (a_k + ix_n) \cdot \prod_{j \neq n}^{\mathcal{N}} ((x_n - i)^2 - y_j) = \prod_{k=1}^4 (a_k - ix_n) \cdot \prod_{j \neq n}^{\mathcal{N}} ((x_n + i)^2 - y_j). \quad (5.17)$$

All the non- q polynomials in this section can be obtained from the Wilson by reductions.

5.2.2 continuous dual Hahn

This is a restricted case of the Wilson polynomial with $a_4 = 0$. The parameters are restricted by $\{a_1^*, a_2^*, a_3^*\} = \{a_1, a_2, a_3\}$, as a set and the various data are [6, 13]:

$$V(x) = \frac{(a_1 + ix)(a_2 + ix)(a_3 + ix)}{2ix(2ix + 1)}, \quad V^*(x) = V(-x), \quad \text{Re } a_j > 0, \quad (5.18)$$

$$\phi_0(x)^2 = \frac{\prod_{j=1}^3 \Gamma(a_j + ix)\Gamma(a_j - ix)}{\Gamma(2ix)\Gamma(-2ix)}, \quad \mathcal{E}(n) = n, \quad (5.19)$$

$$P_n(\eta(x)) = (a_1 + a_2)_n(a_1 + a_3)_n \times {}_3F_2\left(\begin{matrix} -n, a_1 + ix, a_1 - ix \\ a_1 + a_2, a_1 + a_3 \end{matrix} \middle| 1\right). \quad (5.20)$$

The polynomial equation for P_N (3.16) provides algebraic equations for the zeros $\{y_n\}$:

$$\prod_{k=1}^3 (a_k + ix_n) \cdot \prod_{j \neq n}^{\mathcal{N}} ((x_n - i)^2 - y_j) = \prod_{k=1}^3 (a_k - ix_n) \cdot \prod_{j \neq n}^{\mathcal{N}} ((x_n + i)^2 - y_j). \quad (5.21)$$

5.3 Polynomials having $\eta(x) = \cos x$, $0 < x < \pi$, $e^\gamma = q$

The Askey-Wilson polynomial and its six reduced case polynomials belong to this group. For this group, let us introduce new symbols related with the zeros $\{x_n\}$, $(\dot{\eta}(x) = -\sin x)$:

$$z_n \stackrel{\text{def}}{=} e^{ix_n} = \cos x_n + i \sin x_n = y_n - i\dot{\eta}(x_n), \quad n = 1, \dots, \mathcal{N}. \quad (5.22)$$

The matrix \mathcal{M} (3.9) for this group reads

$$\begin{aligned} \mathcal{M}_{nm} = & \delta_{nm} \left\{ \mathcal{E}(\mathcal{N}) + V(x_n) + V^*(x_n) \right. \\ & - \frac{1}{\prod_{j \neq n}^{\mathcal{N}} (y_n - y_j)} \left(V(x_n) \prod_{j \neq n}^{\mathcal{N}} ((qz_n + q^{-1}z_n^{-1})/2 - y_j) \right. \\ & \quad \left. + V^*(x_n) \prod_{j \neq n}^{\mathcal{N}} ((q^{-1}z_n + qz_n^{-1})/2 - y_j) \right) \Big\} \\ & + (1 - \delta_{nm}) \frac{\sin 2x_m}{\sin 2x_n} \frac{(q^{-1} - 1)}{2 \prod_{j \neq n}^{\mathcal{N}} (y_n - y_j)} \\ & \times \left(V(x_n) z_n^{-1} (1 - qz_n^2) \prod_{j \neq n, m}^{\mathcal{N}} ((qz_n + q^{-1}z_n^{-1})/2 - y_j) \right. \\ & \quad \left. + V^*(x_n) z_n (1 - qz_n^{-2}) \prod_{j \neq n, m}^{\mathcal{N}} ((q^{-1}z_n + qz_n^{-1})/2 - y_j) \right). \end{aligned} \quad (5.23)$$

It has eigenvalues $\mathcal{E}(\mathcal{N}) - \mathcal{E}(m)$, $m = 0, 1, \dots, \mathcal{N} - 1$, for arbitrary distinct complex values of $\{x_n\}$ except for the poles of V and V^* and $\{y_n = \cos x_n\}$. For the zeros $\{y_n = \cos x_n\}$ of $P_{\mathcal{N}}$, $P_{\mathcal{N}}(y_n) = 0$, it is straightforward to verify Theorem 3.2 numerically for lower \mathcal{N} . That is, the eigenvectors of the matrix \mathcal{M} (5.23) generate the lower degree polynomials $\{P_m(\eta)\}$, $m = 0, 1, \dots, \mathcal{N} - 1$ as in (3.12).

5.3.1 Askey-Wilson

The Askey-Wilson polynomial is the most general one with the maximal number of four real parameters, or two complex parameters, $\{a_1^*, a_2^*, a_3^*, a_4^*\} = \{a_1, a_2, a_3, a_4\}$ as a set, on top of q . All the other polynomials in this group are obtained by restricting the parameters a_1, \dots, a_4 , in one way or another. The various data are [6, 13]:

$$V(x) = \frac{(1 - a_1 e^{ix})(1 - a_2 e^{ix})(1 - a_3 e^{ix})(1 - a_4 e^{ix})}{(1 - e^{2ix})(1 - qe^{2ix})}, \quad V^*(x) = V(-x), \quad |a_j| < 1, \quad (5.24)$$

$$\phi_0(x)^2 = \frac{(e^{2ix}; q)_\infty (e^{-2ix}; q)_\infty}{\prod_{j=1}^4 (a_j e^{ix}; q)_\infty (a_j e^{-ix}; q)_\infty}, \quad \mathcal{E}(n) = (q^{-n} - 1)(1 - b_4 q^{n-1}), \quad (5.25)$$

$$b_4 \stackrel{\text{def}}{=} a_1 a_2 a_3 a_4, \quad (5.26)$$

$$P_n(\eta(x)) = a_1^{-n} (a_1 a_2, a_1 a_3, a_1 a_4; q)_n \times {}_4\phi_3 \left(\begin{matrix} q^{-n}, b_4 q^{n-1}, a_1 e^{ix}, a_1 e^{-ix} \\ a_1 a_2, a_1 a_3, a_1 a_4 \end{matrix} \middle| q; q \right). \quad (5.27)$$

The polynomial equation for $P_{\mathcal{N}}$ (3.16) provides algebraic equations for the zeros $\{y_n\}$:

$$\begin{aligned} & \prod_{k=1}^4 (1 - a_k z_n) \cdot z_n^{-2} \prod_{j \neq n}^{\mathcal{N}} ((q z_n + q^{-1} z_n^{-1})/2 - y_j) \\ &= \prod_{k=1}^4 (1 - a_k z_n^{-1}) \cdot z_n^2 \prod_{j \neq n}^{\mathcal{N}} ((q^{-1} z_n + q z_n^{-1})/2 - y_j). \end{aligned} \quad (5.28)$$

5.3.2 Continuous dual q -Hahn

The continuous dual q -Hahn polynomial is obtained by restricting $a_4 = 0$ in the Askey-Wilson polynomial §5.3.1. The various data are [6, 13]:

$$V(x) = \frac{(1 - a_1 e^{ix})(1 - a_2 e^{ix})(1 - a_3 e^{ix})}{(1 - e^{2ix})(1 - q e^{2ix})}, \quad V^*(x) = V(-x), \quad |a_j| < 1, \quad (5.29)$$

$$\phi_0(x)^2 = \frac{(e^{2ix}; q)_\infty (e^{-2ix}; q)_\infty}{\prod_{j=1}^3 (a_j e^{ix}; q)_\infty (a_j e^{-ix}; q)_\infty}, \quad \mathcal{E}(n) = q^{-n} - 1, \quad (5.30)$$

$$\{a_1^*, a_2^*, a_3^*\} = \{a_1, a_2, a_3\} \quad (\text{as a set}), \quad (5.31)$$

$$P_n(\eta(x)) = a_1^{-n} (a_1 a_2, a_1 a_3; q)_n \times {}_3\phi_2 \left(\begin{matrix} q^{-n}, a_1 e^{ix}, a_1 e^{-ix} \\ a_1 a_2, a_1 a_3 \end{matrix} \middle| q; q \right). \quad (5.32)$$

5.3.3 Al-Salam-Chihara

This is a further restriction of the continuous dual q -Hahn polynomial §5.3.2 with $a_3 = 0$. The various data are [6, 13]:

$$V(x) = \frac{(1 - a_1 e^{ix})(1 - a_2 e^{ix})}{(1 - e^{2ix})(1 - q e^{2ix})}, \quad V^*(x) = V(-x), \quad |a_j| < 1, \quad (5.33)$$

$$\phi_0(x)^2 = \frac{(e^{2ix}; q)_\infty (e^{-2ix}; q)_\infty}{\prod_{j=1}^2 (a_j e^{ix}; q)_\infty (a_j e^{-ix}; q)_\infty}, \quad \mathcal{E}(n) = q^{-n} - 1, \quad (5.34)$$

$$\{a_1^*, a_2^*\} = \{a_1, a_2\} \quad (\text{as a set}), \quad (5.35)$$

$$P_n(\eta(x)) = a_1^{-n} (a_1 a_2; q)_n \times {}_3\phi_2 \left(\begin{matrix} q^{-n}, a_1 e^{ix}, a_1 e^{-ix} \\ a_1 a_2, 0 \end{matrix} \middle| q; q \right). \quad (5.36)$$

5.3.4 Continuous big q -Hermite

This is a further restriction of the Al-Salam-Chihara polynomial §5.3.3 with $a_2 = 0$ and it depends on one real parameter a on top of q . The various data are [6, 13]:

$$V(x) = \frac{(1 - a e^{ix})}{(1 - e^{2ix})(1 - q e^{2ix})}, \quad V^*(x) = V(-x), \quad -1 < a < 1, \quad (5.37)$$

$$\phi_0(x)^2 = \frac{(e^{2ix}; q)_\infty (e^{-2ix}; q)_\infty}{(a e^{ix}; q)_\infty (a e^{-ix}; q)_\infty}, \quad \mathcal{E}(n) = q^{-n} - 1, \quad (5.38)$$

$$P_n(\eta(x)) = a^{-n} \times {}_3\phi_2 \left(\begin{matrix} q^{-n}, a e^{ix}, a e^{-ix} \\ 0, 0 \end{matrix} \middle| q; q \right). \quad (5.39)$$

5.3.5 Continuous q -Hermite

This is a q analogue of the Hermite polynomial, depending on q only. It also provides the simplest dynamical realisation of the q -oscillator algebra [17]. The various data are [6, 13]:

$$V(x) = \frac{1}{(1 - e^{2ix})(1 - q e^{2ix})}, \quad V^*(x) = V(-x), \quad (5.40)$$

$$\phi_0(x)^2 = (e^{2ix}; q)_\infty (e^{-2ix}; q)_\infty, \quad \mathcal{E}(n) = q^{-n} - 1, \quad (5.41)$$

$$P_n(\eta(x)) = e^{inx} {}_2\phi_0 \left(\begin{matrix} q^{-n}, 0 \\ - \end{matrix} \middle| q; q^n e^{-2ix} \right). \quad (5.42)$$

The polynomial equation for P_N (3.16) provides simple algebraic equations for the zeros $\{y_n\}$:

$$z_n^{-2} \prod_{j \neq n}^N ((q z_n + q^{-1} z_n^{-1})/2 - y_j) = z_n^2 \prod_{j \neq n}^N ((q^{-1} z_n + q z_n^{-1})/2 - y_j). \quad (5.43)$$

5.3.6 Continuous q -Jacobi

This polynomial depends on two real parameters α and β and other data are [6, 13]:

$$V(x) = \frac{(1 - q^{\frac{1}{2}(\alpha + \frac{1}{2})} e^{ix})(1 - q^{\frac{1}{2}(\alpha + \frac{3}{2})} e^{ix})(1 + q^{\frac{1}{2}(\beta + \frac{1}{2})} e^{ix})(1 + q^{\frac{1}{2}(\beta + \frac{3}{2})} e^{ix})}{(1 - e^{2ix})(1 - q e^{2ix})}, \quad (5.44)$$

$$V^*(x) = V(-x), \quad \alpha, \beta \geq -\frac{1}{2}, \quad \mathcal{E}(n) = (q^{-n} - 1)(1 - q^{n+\alpha+\beta+1}), \quad (5.45)$$

$$\phi_0(x)^2 = \frac{(e^{2ix}; q)_\infty (e^{-2ix}; q)_\infty}{(q^{\frac{1}{2}(\alpha + \frac{1}{2})} e^{ix}, -q^{\frac{1}{2}(\beta + \frac{1}{2})} e^{ix}; q^{\frac{1}{2}})_\infty (q^{\frac{1}{2}(\alpha + \frac{1}{2})} e^{-ix}, -q^{\frac{1}{2}(\beta + \frac{1}{2})} e^{-ix}; q^{\frac{1}{2}})_\infty}, \quad (5.46)$$

$$P_n(\eta(x)) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} {}_4\phi_3 \left(\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1}, q^{\frac{1}{2}(\alpha + \frac{1}{2})} e^{ix}, q^{\frac{1}{2}(\alpha + \frac{1}{2})} e^{-ix} \\ q^{\alpha+1}, -q^{\frac{1}{2}(\alpha + \beta + 1)}, -q^{\frac{1}{2}(\alpha + \beta + 2)} \end{matrix} \middle| q; q \right). \quad (5.47)$$

5.3.7 Continuous q -Laguerre

This is a further restriction ($\beta \rightarrow \infty$ or $q^\beta \rightarrow 0$) of the continuous q -Jacobi polynomial §5.3.6. Many formulas are simplified [6, 13]:

$$V(x) = \frac{(1 - q^{\frac{1}{2}(\alpha + \frac{1}{2})}e^{ix})(1 - q^{\frac{1}{2}(\alpha + \frac{3}{2})}e^{ix})}{(1 - e^{2ix})(1 - qe^{2ix})}, \quad V^*(x) = V(-x), \quad \alpha \geq -\frac{1}{2}, \quad (5.48)$$

$$\phi_0(x)^2 = \frac{(e^{2ix}; q)_\infty (e^{-2ix}; q)_\infty}{(q^{\frac{1}{2}(\alpha + \frac{1}{2})}e^{ix}; q^{\frac{1}{2}})_\infty (q^{\frac{1}{2}(\alpha + \frac{1}{2})}e^{-ix}; q^{\frac{1}{2}})_\infty}, \quad \mathcal{E}(n) = q^{-n} - 1, \quad (5.49)$$

$$P_n(\eta(x)) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} {}_3\phi_2 \left(\begin{matrix} q^{-n}, q^{\frac{1}{2}(\alpha + \frac{1}{2})}e^{ix}, q^{\frac{1}{2}(\alpha + \frac{1}{2})}e^{-ix} \\ q^{\alpha+1}, 0 \end{matrix} \middle| q; q \right). \quad (5.50)$$

6 Examples from discrete quantum mechanics with real shifts

The polynomials belonging to this class are also called *classical orthogonal polynomials of a discrete variable* [18]. The second order difference operator $\tilde{\mathcal{H}}$ governing these polynomials (2.6) reads

$$\tilde{\mathcal{H}} = B(x)(1 - e^\partial) + D(x)(1 - e^{-\partial}). \quad (6.1)$$

Here two non-negative functions $B(x) \geq 0, D(x) \geq 0$ of x are defined on non-negative integer lattice points, finite $[0, 1, \dots, N]$, or infinite $[0, 1, \dots, \infty)$, with the boundary conditions

$$D(0) = 0, \quad B(N) = 0, \quad \mathcal{N} < N, \quad (6.2)$$

in which the latter two conditions apply only to the finite lattice cases. The shift operators $e^{\pm\partial}$ act on a function defined on the above non-negative integer lattice points and shift it either ± 1 :

$$e^{\pm\partial}\psi(x) = \psi(x \pm 1).$$

The polynomials are assembled into four groups according to the form of the sinusoidal coordinate $\eta(x) = x, x(x + d)$ and $q^{\pm x}$ -linear and $q^{\pm x}$ -bilinear. In each group, the finite lattice cases are followed by infinite lattice ones. The most generic member will be followed by simpler ones. For a comprehensive exposition of these polynomials in the discrete quantum mechanics formulation and their applications, see [12, 14, 19].

6.1 Polynomials having $\eta(x) = x$, $[0, 1, \dots, N]$ or $[0, 1, \dots, \infty)$

This group consists of four polynomials, the Hahn §6.1.1, the Krawtchouk §6.1.2 (on finite lattices) and the Meixner §6.1.3 and the Charlier §6.1.4, both on infinite lattices. The matrix \mathcal{M} (3.9) for this group reads

$$\begin{aligned} \mathcal{M}_{nm} = & \delta_{nm} \left(\frac{B(x_n) \prod_{j \neq n}^{\mathcal{N}} (x_n + 1 - x_j) + D(x_n) \prod_{j \neq n}^{\mathcal{N}} (x_n - 1 - x_j)}{\prod_{j \neq n}^{\mathcal{N}} (x_n - x_j)} \right. \\ & \left. + \mathcal{E}(\mathcal{N}) - B(x_n) - D(x_n) \right) \\ & + (1 - \delta_{nm}) \frac{\left(B(x_n) \prod_{j \neq n, m}^{\mathcal{N}} (x_n + 1 - x_j) - D(x_n) \prod_{j \neq n, m}^{\mathcal{N}} (x_n - 1 - x_j) \right)}{\prod_{j \neq n}^{\mathcal{N}} (x_n - x_j)}. \end{aligned} \quad (6.3)$$

It has eigenvalues $\mathcal{E}(\mathcal{N}) - \mathcal{E}(m)$, $m = 0, 1, \dots, \mathcal{N} - 1$, for arbitrary distinct complex values of $\{x_n\}$. For the zeros $\{x_n\}$ of $P_{\mathcal{N}}$, $P_{\mathcal{N}}(x_n) = 0$, it is straightforward to verify Theorem 3.2 numerically for lower \mathcal{N} . That is, the eigenvectors of the matrix \mathcal{M} (6.3) generate the lower degree polynomials $\{P_m(x)\}$, $m = 0, 1, \dots, \mathcal{N} - 1$ as in (3.12).

For these polynomials, the matrix \mathcal{M} (6.3) simplifies by using the algebraic equations for the zeros $\{x_n\}$:

$$B(x_n) \prod_{j \neq n}^{\mathcal{N}} (x_n + 1 - x_j) = D(x_n) \prod_{j \neq n}^{\mathcal{N}} (x_n - 1 - x_j), \quad (6.4)$$

$$\begin{aligned} \mathcal{M}_{nm} = & \delta_{nm} \left(B(x_n) \prod_{j \neq n} \left(1 + \frac{1}{x_n - x_j} \right) + \mathcal{E}(\mathcal{N}) - B(x_n) - D(x_n) \right) \\ & + (1 - \delta_{nm}) B(x_n) \prod_{j \neq n} \left(1 + \frac{1}{x_n - x_j} \right) \cdot \left(\frac{1}{x_n + 1 - x_m} - \frac{1}{x_n - 1 - x_m} \right). \end{aligned} \quad (6.5)$$

6.1.1 Hahn

This polynomial depends on two real parameters a and b besides N [6, 12]:

$$B(x) = (x + a)(N - x), \quad D(x) = x(b + N - x), \quad a > 0, \quad b > 0, \quad (6.6)$$

$$\phi_0(x)^2 = \frac{N!}{x!(N-x)!} \frac{(a)_x (b)_{N-x}}{(b)_N}, \quad \mathcal{E}(n) = n(n + a + b - 1), \quad (6.7)$$

$$P_n(x) = {}_3F_2 \left(\begin{matrix} -n, n + a + b - 1, -x \\ a, -N \end{matrix} \middle| 1 \right). \quad (6.8)$$

6.1.2 Krawtchouk

This polynomial depends on one real parameter p besides N [6, 12]:

$$B(x) = p(N - x), \quad D(x) = (1 - p)x, \quad 0 < p < 1, \quad \mathcal{E}(n) = n, \quad (6.9)$$

$$\phi_0(x)^2 = \frac{N!}{x!(N-x)!} \left(\frac{p}{1-p} \right)^x, \quad P_n(x) = {}_2F_1 \left(\begin{matrix} -n, -x \\ -N \end{matrix} \middle| p^{-1} \right). \quad (6.10)$$

This polynomial (6.10) is symmetric under the interchange $x \leftrightarrow n$, with $\mathcal{E}(n) = n$ and $\eta(x) = x$.

6.1.3 Meixner

This is another example of the self-dual polynomial (6.12), which is symmetric under the interchange $x \leftrightarrow n$, with $\mathcal{E}(n) = n$ and $\eta(x) = x$. This polynomial depends on two real parameters β and c , [6, 12]:

$$B(x) = \frac{c}{1-c}(x + \beta), \quad D(x) = \frac{1}{1-c}x, \quad \beta > 0, \quad 0 < c < 1, \quad (6.11)$$

$$\phi_0(x)^2 = \frac{(\beta)_x c^x}{x!}, \quad P_n(x) = {}_2F_1 \left(\begin{matrix} -n, -x \\ \beta \end{matrix} \middle| 1 - c^{-1} \right), \quad \mathcal{E}(n) = n. \quad (6.12)$$

6.1.4 Charlier

The Charlier polynomial depends on one real parameter a and is also self-dual $x \leftrightarrow n$ [6, 12]:

$$B(x) = a, \quad D(x) = x, \quad a > 0, \quad \mathcal{E}(n) = n, \quad (6.13)$$

$$\phi_0(x)^2 = \frac{a^x}{x!}, \quad P_n(x) = {}_2F_0 \left(\begin{matrix} -n, -x \\ - \end{matrix} \middle| -a^{-1} \right). \quad (6.14)$$

6.2 Polynomials having $\eta(x)$ quadratic in x

This group consists of two polynomials, the Racah §6.2.1 and the dual Hahn §6.2.2, both on finite lattices. The matrix \mathcal{M} (3.9) for this group reads

$$\begin{aligned} \mathcal{M}_{nm} = \delta_{nm} & \left(\frac{B(x_n) \prod_{j \neq n}^{\mathcal{N}} (\eta(x_n + 1) - y_j) + D(x_n) \prod_{j \neq n}^{\mathcal{N}} (\eta(x_n - 1) - y_j)}{\prod_{j \neq n}^{\mathcal{N}} (y_n - y_j)} \right. \\ & \left. + \mathcal{E}(\mathcal{N}) - B(x_n) - D(x_n) \right) \\ & + (1 - \delta_{nm}) \frac{\dot{\eta}(x_m)}{\dot{\eta}(x_n)} \frac{1}{\prod_{j \neq n}^{\mathcal{N}} (y_n - y_j)} \left(B(x_n) \prod_{j \neq m}^{\mathcal{N}} (\eta(x_n + 1) - y_j) \right. \end{aligned}$$

$$+D(x_n) \prod_{j \neq m}^{\mathcal{N}} (\eta(x_n - 1) - y_j) \Bigg). \quad (6.15)$$

It has eigenvalues $\mathcal{E}(\mathcal{N}) - \mathcal{E}(m)$, $m = 0, 1, \dots, \mathcal{N} - 1$, for arbitrary distinct complex values of $\{x_n\}$ except for the poles of $B(x)$ and $D(x)$. For the zeros $\{y_n = \eta(x_n)\}$ of $P_{\mathcal{N}}$, $P_{\mathcal{N}}(y_n) = 0$, it is straightforward to verify Theorem 3.2 numerically for lower \mathcal{N} . That is the eigenvectors of the matrix \mathcal{M} (6.3) generate the lower degree polynomials $\{P_m(\eta)\}$, $m = 0, 1, \dots, \mathcal{N} - 1$ as in (3.12).

6.2.1 Racah

We adopt a parametrisation [12] designed to reveal the symmetry of the difference Racah equation. This polynomial has four real parameters, a , b , c and d , one of which must be equal to $-N$. Here we choose the parameter ranges

$$c = -N, \quad d > 0, \quad a > N + d, \quad 0 < b < 1 + d. \quad (6.16)$$

The various data are [6, 12]:

$$\begin{aligned} B(x) &= -\frac{(x+a)(x+b)(x+c)(x+d)}{(2x+d)(2x+1+d)}, \\ D(x) &= -\frac{(x+d-a)(x+d-b)(x+d-c)x}{(2x-1+d)(2x+d)}, \end{aligned} \quad (6.17)$$

$$\mathcal{E}(n) = n(n + \tilde{d}), \quad \tilde{d} \stackrel{\text{def}}{=} a + b + c - d - 1, \quad (6.18)$$

$$\eta(x) = x(x + d), \quad \dot{\eta}(x) = 2x + d, \quad (6.19)$$

$$\phi_0(x)^2 = \frac{(a, b, c, d)_x}{(1 + d - a, 1 + d - b, 1 + d - c, 1)_x} \frac{2x + d}{d}, \quad (6.20)$$

$$P_n(\eta(x)) = {}_4F_3 \left(\begin{matrix} -n, n + \tilde{d}, -x, x + d \\ a, b, c \end{matrix} \middle| 1 \right). \quad (6.21)$$

All the non- q polynomials in this section can be obtained from the Racah by reductions.

6.2.2 dual Hahn

We adopt the parametrisation of the dual Hahn polynomial so that the duality ($x \leftrightarrow n$) with the Hahn polynomial §6.1.1 is obvious. Thus the parameters (a, b) are different from the standard ones [6] for the dual Hahn polynomial:

$$B(x) = \frac{(x+a)(x+a+b-1)(N-x)}{(2x-1+a+b)(2x+a+b)}, \quad a > 0, \quad b > 0, \quad (6.22)$$

$$D(x) = \frac{x(x+b-1)(x+a+b+N-1)}{(2x-2+a+b)(2x-1+a+b)}, \quad (6.23)$$

$$\mathcal{E}(n) = n, \quad \eta(x) = x(x+a+b-1), \quad \dot{\eta}(x) = 2x+a+b-1, \quad (6.24)$$

$$\phi_0(x)^2 = \frac{N!}{x!(N-x)!} \frac{(a)_x (2x+a+b-1)(a+b)_N}{(b)_x (x+a+b-1)_{N+1}}, \quad (6.25)$$

$$P_n(\eta(x)) = {}_3F_2 \left(\begin{matrix} -n, x+a+b-1, -x \\ a, -N \end{matrix} \middle| 1 \right). \quad (6.26)$$

6.3 Polynomials having $\eta(x)$ linear in q^{-x} , $[0, 1, \dots, N]$ or $[0, 1, \dots, \infty)$

This group of polynomials have the sinusoidal coordinate $\eta(x) = q^{-x} - 1$. Seven polynomials belong to this group; q -Hahn §6.3.1, quantum q -Krawtchouk §6.3.2, q -Krawtchouk §6.3.3, affine q -Krawtchouk §6.3.4, q -Meixner §6.3.5, Al-Salam-Carlitz II §6.3.6 and q -Charlier §6.3.7. As for the naming of the polynomials we follow [6]. The first four examples are on finite lattices. The remaining three are on infinite lattices. The matrix \mathcal{M} (3.9) for this group has essentially the same structure and properties as those for the Racah (6.15).

6.3.1 q -Hahn

This polynomial depends on two real parameters a and b . We choose the parameter range $0 < a < 1$ and $0 < b < 1$. Other data are [12]:

$$B(x) = (1 - aq^x)(q^{x-N} - 1), \quad D(x) = aq^{-1}(1 - q^x)(q^{x-N} - b), \quad (6.27)$$

$$\phi_0(x)^2 = \frac{(q; q)_N}{(q; q)_x (q; q)_{N-x}} \frac{(a; q)_x (b; q)_{N-x}}{(b; q)_N a^x}, \quad \mathcal{E}(n) = (q^{-n} - 1)(1 - abq^{n-1}), \quad (6.28)$$

$$P_n(\eta(x)) = {}_3\phi_2 \left(\begin{matrix} q^{-n}, abq^{n-1}, q^{-x} \\ a, q^{-N} \end{matrix} \middle| q; q \right). \quad (6.29)$$

6.3.2 quantum q -Krawtchouk

This polynomial depends on one parameter $p > q^{-N}$:

$$B(x) = p^{-1}q^x(q^{x-N} - 1), \quad D(x) = (1 - q^x)(1 - p^{-1}q^{x-N-1}), \quad \mathcal{E}(n) = 1 - q^n, \quad (6.30)$$

$$\phi_0(x)^2 = \frac{(q; q)_N}{(q; q)_x (q; q)_{N-x}} \frac{p^{-x}q^{x(x-1-N)}}{(p^{-1}q^{-N}; q)_x}, \quad P_n(\eta(x)) = {}_2\phi_1 \left(\begin{matrix} q^{-n}, q^{-x} \\ q^{-N} \end{matrix} \middle| q; pq^{n+1} \right). \quad (6.31)$$

6.3.3 q -Krawtchouk

This polynomial depends on one positive parameter $p > 0$ [6, 12]:

$$B(x) = q^{x-N} - 1, \quad D(x) = p(1 - q^x), \quad \mathcal{E}(n) = (q^{-n} - 1)(1 + pq^n), \quad (6.32)$$

$$\phi_0(x)^2 = \frac{(q; q)_N}{(q; q)_x (q; q)_{N-x}} p^{-x} q^{\frac{1}{2}x(x-1)-xN}, \quad P_n(\eta(x)) = {}_3\phi_2 \left(\begin{matrix} q^{-n}, q^{-x}, -pq^n \\ q^{-N}, 0 \end{matrix} \middle| q; q \right). \quad (6.33)$$

6.3.4 affine q -Krawtchouk

This polynomial has one positive parameter $0 < p < q^{-1}$ and it is self-dual ($x \leftrightarrow n$):

$$B(x) = (q^{x-N} - 1)(1 - pq^{x+1}), \quad D(x) = pq^{x-N}(1 - q^x), \quad \mathcal{E}(n) = q^{-n} - 1, \quad (6.34)$$

$$\phi_0(x)^2 = \frac{(q; q)_N}{(q; q)_x (q; q)_{N-x}} \frac{(pq; q)_x}{(pq)^x}, \quad P_n(\eta(x)) = {}_3\phi_2 \left(\begin{matrix} q^{-n}, q^{-x}, 0 \\ pq, q^{-N} \end{matrix} \middle| q; q \right). \quad (6.35)$$

6.3.5 q -Meixner

This polynomial has two real parameters b and c with $0 < b < q^{-1}$, $c > 0$ and other data are [6, 12]:

$$B(x) = cq^x(1 - bq^{x+1}), \quad D(x) = (1 - q^x)(1 + bcq^x), \quad \mathcal{E}(n) = 1 - q^n, \quad (6.36)$$

$$\phi_0(x)^2 = \frac{(bq; q)_x}{(q, -bcq; q)_x} c^x q^{\frac{1}{2}x(x-1)}, \quad P_n(\eta(x)) = {}_2\phi_1 \left(\begin{matrix} q^{-n}, q^{-x} \\ bq \end{matrix} \middle| q; -c^{-1}q^{n+1} \right). \quad (6.37)$$

6.3.6 Al-Salam-Carlitz II

The polynomial depends on one real parameter $0 < a < q^{-1}$ [6, 12]:

$$B(x) = aq^{2x+1}, \quad D(x) = (1 - q^x)(1 - aq^x), \quad \mathcal{E}(n) = 1 - q^n, \quad (6.38)$$

$$\phi_0(x)^2 = \frac{a^x q^{x^2}}{(q, aq; q)_x}, \quad P_n(\eta(x)) = {}_2\phi_0 \left(\begin{matrix} q^{-n}, q^{-x} \\ - \end{matrix} \middle| q; a^{-1}q^n \right). \quad (6.39)$$

6.3.7 q -Charlier

The polynomial depends on one positive parameter $a > 0$ [6, 12]:

$$B(x) = aq^x, \quad D(x) = 1 - q^x, \quad \mathcal{E}(n) = 1 - q^n, \quad (6.40)$$

$$\phi_0(x)^2 = \frac{a^x q^{\frac{1}{2}x(x-1)}}{(q; q)_x}, \quad P_n(\eta(x)) = {}_2\phi_1 \left(\begin{matrix} q^{-n}, q^{-x} \\ 0 \end{matrix} \middle| q; -a^{-1}q^{n+1} \right). \quad (6.41)$$

6.4 Polynomials having $\eta(x)$ linear in q^x , $[0, 1, \dots, \infty)$

This group of polynomials have the sinusoidal coordinate $\eta(x) = 1 - q^x$. Three polynomials defined on infinite lattices, little q -Jacobi §6.4.1, little q -Laguerre §6.4.2 and alternative q -Charlier §6.4.3, belong to this group. The matrix \mathcal{M} (3.9) for this group has essentially the same structure and properties as that for the Racah (6.15).

6.4.1 little q -Jacobi

This polynomial depends on two positive parameters $0 < a, b < q^{-1}$ [6, 12]:

$$B(x) = a(q^{-x} - bq), \quad D(x) = q^{-x} - 1, \quad \mathcal{E}(n) = (q^{-n} - 1)(1 - abq^{n+1}), \quad (6.42)$$

$$\phi_0(x)^2 = \frac{(bq; q)_x}{(q; q)_x} (aq)^x, \quad P_n(\eta(x)) = (-a)^{-n} q^{-\frac{1}{2}n(n+1)} \frac{(aq; q)_n}{(bq; q)_n} {}_2\phi_1 \left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} \middle| q; q^{x+1} \right). \quad (6.43)$$

6.4.2 little q -Laguerre/Wall

Putting $b = 0$ in the little q -Jacobi §6.4.1 gives this polynomial [6, 12]:

$$B(x) = aq^{-x}, \quad D(x) = q^{-x} - 1, \quad \mathcal{E}(n) = q^{-n} - 1, \quad (6.44)$$

$$\phi_0(x)^2 = \frac{(aq)^x}{(q; q)_x}, \quad P_n(\eta(x)) = {}_2\phi_0 \left(\begin{matrix} q^{-n}, q^{-x} \\ - \end{matrix} \middle| q; a^{-1}q^x \right). \quad (6.45)$$

6.4.3 alternative q -Charlier

This polynomial depends on one positive parameter $a > 0$ [6, 12]:

$$B(x) = a, \quad D(x) = q^{-x} - 1, \quad \mathcal{E}(n) = (q^{-n} - 1)(1 + aq^n), \quad (6.46)$$

$$\phi_0(x)^2 = \frac{a^x q^{\frac{1}{2}x(x+1)}}{(q; q)_x}, \quad P_n(\eta(x)) = q^{nx} {}_2\phi_1 \left(\begin{matrix} q^{-n}, q^{-x} \\ 0 \end{matrix} \middle| q; -a^{-1}q^{-n+1} \right). \quad (6.47)$$

6.5 Polynomials having $\eta(x)$ bilinear in q^{-x} and q^x , $[0, 1, \dots, N]$

This group of polynomials have the sinusoidal coordinate $\eta(x) = (q^{-x} - 1)(1 - Aq^x)$ with some constant A and on finite lattices. The q -Racah §6.5.1 and dual q -Hahn §6.5.2 belong to this group. The matrix \mathcal{M} (3.9) for this group has essentially the same structure and properties as those for the Racah (6.15).

6.5.1 q -Racah

This polynomial is the most general of all the classical orthogonal polynomials of a discrete variable. The rest of the polynomials in §6 is obtained from q -Racah by reductions. We adopt a parametrisation [12] designed to reveal the symmetry of the difference q -Racah equation. This polynomial has four real parameters, a, b, c and d , one of which must be equal to q^{-N} . The set of parameters is different from the standard one $(\alpha, \beta, \gamma, \delta)$ in the same manner as for the Racah polynomial §6.2.1.

Here we choose the parameter ranges and introduce \tilde{d} :

$$c = q^{-N}, \quad 0 < d < 1, \quad 0 < a < q^N d, \quad qd < b < 1, \quad \tilde{d} \stackrel{\text{def}}{=} abcd^{-1}q^{-1}. \quad (6.48)$$

Other data are [6, 12]:

$$B(x) = -\frac{(1 - aq^x)(1 - bq^x)(1 - cq^x)(1 - dq^x)}{(1 - dq^{2x})(1 - dq^{2x+1})}, \quad (6.49)$$

$$D(x) = -\tilde{d} \frac{(1 - a^{-1}dq^x)(1 - b^{-1}dq^x)(1 - c^{-1}dq^x)(1 - q^x)}{(1 - dq^{2x-1})(1 - dq^{2x})}, \quad (6.50)$$

$$\mathcal{E}(n) = (q^{-n} - 1)(1 - \tilde{d}q^n), \quad \eta(x) = (q^{-x} - 1)(1 - dq^x), \quad (6.51)$$

$$\phi_0(x)^2 = \frac{(a, b, c, d; q)_x}{(a^{-1}dq, b^{-1}dq, c^{-1}dq, q; q)_x} \frac{1 - dq^{2x}}{\tilde{d}^x (1 - d)}, \quad (6.52)$$

$$P_n(\eta(x)) = {}_4\phi_3 \left(\begin{matrix} q^{-n}, \tilde{d}q^n, q^{-x}, dq^x \\ a, b, c \end{matrix} \middle| q; q \right). \quad (6.53)$$

6.5.2 dual q -Hahn

We adopt the same parameters (a, b) ($0 < a, b < 1$) for the q -Hahn §6.3.1 and dual q -Hahn polynomials [6, 12]:

$$B(x) = \frac{(q^{x-N} - 1)(1 - aq^x)(1 - abq^{x-1})}{(1 - abq^{2x-1})(1 - abq^{2x})}, \quad (6.54)$$

$$D(x) = aq^{x-N-1} \frac{(1 - q^x)(1 - abq^{x+N-1})(1 - bq^{x-1})}{(1 - abq^{2x-2})(1 - abq^{2x-1})}, \quad (6.55)$$

$$\mathcal{E}(n) = q^{-n} - 1, \quad \eta(x) = (q^{-x} - 1)(1 - abq^{x-1}), \quad (6.56)$$

$$\phi_0(x)^2 = \frac{(q; q)_N}{(q; q)_x (q; q)_{N-x}} \frac{(a, abq^{-1}; q)_x}{(abq^N, b; q)_x} \frac{1 - abq^{2x-1}}{a^x (1 - abq^{-1})}, \quad (6.57)$$

$$P_n(\eta(x)) = {}_3\phi_2 \left(\begin{matrix} q^{-n}, abq^{x-1}, q^{-x} \\ a, q^{-N} \end{matrix} \middle| q; q \right). \quad (6.58)$$

7 Summary and Comments

Based on the quantum mechanical reformulation of classical orthogonal polynomials [12, 13, 14], an $\mathcal{N} \times \mathcal{N}$ matrix \mathcal{M} (3.9) describing the small oscillations around the zeros of the degree \mathcal{N} polynomial is derived. By construction, its components depend explicitly on the zeros, but its eigenvalues (3.10) are independent of them. Its eigenvalues are the difference of those of the differential/difference operator $\tilde{\mathcal{H}}$ (2.5), which governs the classical orthogonal polynomial, corresponding to the degree \mathcal{N} and a lower degree, Theorem 3.1.

The corresponding eigenvectors (3.11) of \mathcal{M} provide the representations of the lower degree polynomials in terms of the zeros of the degree \mathcal{N} polynomial, (3.12), Theorem 3.2. It should be stressed that these theorems are valid *universally for all the classical orthogonal polynomials*. We have provided the necessary data for most of the classical orthogonal polynomials ranging from the Hermite, Laguerre, Jacobi, Wilson, Askey-Wilson, Racah, q -Racah and their reduced form polynomials for self contained verification of the main results. The data include the proper ranges of the parameters, which are important for numerical verification.

The ingredients of the matrix \mathcal{M} (3.9) are the *sinusoidal coordinate* $\eta(x)$ (2.4) and the differential/difference operator $\tilde{\mathcal{H}}$, that is, the analytic functions $V(x)$ and $V^*(x)$ (5.1) for the Wilson, Askey-Wilson and their reduced form polynomials, and the two non-negative functions $B(x)$ and $D(x)$ (6.1) for the Racah, q -Racah and their reduced form polynomials. The close relationship between these functions and the sinusoidal coordinates was elucidated in [20].

The present research was inspired by a recent work of Bihun and Calogero [9] discussing the properties of the zeros of the polynomials belonging to the Askey scheme. Their paper is a certain generalisation of the old results by Calogero and his co-authors [16] on the properties of the zeros of the Hermite, Laguerre and Jacobi polynomials. Although our matrix \mathcal{M} (3.9) and their matrices in [9, 16] have related eigenvalues, our \mathcal{M} (3.9) is conceptually and structurally different from those matrices in the two papers [9, 16]. The motivation and guiding principle of the earlier works [1, 2, 9, 16], [21]–[26] were the Diophantine properties of the Hessian matrices describing the small oscillations around the equilibrium of *exactly solvable multi-particle dynamics*. In contrast, our matrix \mathcal{M} (3.9) describes the perturbations around the zeros of a classical orthogonal polynomial of a *single variable*.

For the Classical orthogonal polynomials, *i.e.* the Hermite, Laguerre and Jacobi polynomials, we have demonstrated algebraic derivation of the Theorems in §4 with the help of the old results in [16]. These contents can be reformulated by using ‘finite dimensional representations of differential operators’, which was developed by Calogero [27]. It is a good challenge to deliver similar algebraic derivation of the **Corollary 3.4** (3.15) for each of the polynomials in the Askey scheme.

After completing this paper, we noticed a recent publication [28] discussing related matrices for q -Askey scheme polynomials and another discussing finite dimensional representations

of difference operators [29]. The latter might be useful for algebraic derivation of the **Corollary 3.4** (3.15) for the $(q-)$ Askey scheme of hypergeometric orthogonal polynomials.

Acknowledgements

It is a pleasure to thank the organizers of the CRM-ICMAT Workshop on “Exceptional orthogonal polynomials and exact solutions in mathematical physics” (Segovia, Spain, 7–12 July 2014). R. S. thanks Francesco Calogero and Kazuhiko Aomoto for useful and insightful discussion and comments. He thanks Pauchy Hwang and Department of Physics, National Taiwan University for hospitality.

Appendix: Symbols and definitions

For self-containedness we collect several definitions related to the $(q-)$ hypergeometric functions [6].

◦ Shifted factorial $(a)_n$:

$$(a)_n \stackrel{\text{def}}{=} \prod_{k=1}^n (a+k-1) = a(a+1) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}. \quad (\text{A.1})$$

◦ q -Shifted factorial $(a; q)_n$:

$$(a; q)_n \stackrel{\text{def}}{=} \prod_{k=1}^n (1 - aq^{k-1}) = (1-a)(1-aq) \cdots (1-aq^{n-1}). \quad (\text{A.2})$$

◦ hypergeometric functions ${}_rF_s$:

$${}_rF_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| z \right) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r)_n}{(b_1, \dots, b_s)_n} \frac{z^n}{n!}, \quad (\text{A.3})$$

where $(a_1, \dots, a_r)_n \stackrel{\text{def}}{=} \prod_{j=1}^r (a_j)_n = (a_1)_n \cdots (a_r)_n$.

◦ q -hypergeometric functions (the basic hypergeometric functions) ${}_r\phi_s$:

$${}_r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q; z \right) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n} (-1)^{(1+s-r)n} q^{(1+s-r)n(n-1)/2} \frac{z^n}{(q; q)_n}, \quad (\text{A.4})$$

where $(a_1, \dots, a_r; q)_n \stackrel{\text{def}}{=} \prod_{j=1}^r (a_j; q)_n = (a_1; q)_n \cdots (a_r; q)_n$.

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